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CONSTITUTIVE EQUATIONS FOR ELASTIC–PLASTIC MATERIALS AT FINITE STRAIN

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Abstract—Constitutive equations are suggested for describing the behavior of elastic–plastic materials undergoing large strains. A special kinematical viewpoint is taken, so that the elastic and plastic deformation processes can be considered separately. This separation is also accommodated by a simplified thermodynamical theory of the deformation process. The elastic constitutive equation is written as a rate equation, after examining the interpretation of elastic isotropy in view of the particular kinematical description employed. To describe plastic deformation, a rate equation, which exhibits no dependence on the rate at which previous states have been traversed, is suggested. After the general relations have been put in appropriate form some simplifications based on physical assumptions are considered. The physical assumptions are based on the behavior of metals under large stress, high speed loading, such as in the penetration problem. Under these operating conditions, the thermoelastic effects dominate and plasticity plays a minor role. Consequently, a simple model of plastic deformation usually suffices. The analysis is presented in direct (matrix) notation and is valid for arbitrary stress states.

INTRODUCTION

DURING elastic–plastic deformation processes of metals, under most conditions of operation, either the maximum elastic and plastic strains achieved are infinitesimal and of roughly the same order (about 10^{-3}), or the plastic strains become so much larger than the elastic strains that the plastic strains must be considered finite and the elastic strains may be neglected. In certain applications, however, both elastic and plastic strains must be considered finite. It then becomes necessary to construct constitutive models for elastic–plastic materials undergoing finite deformation.

A fact which must be kept in mind in the formulation of constitutive equations is that the most one can expect of any mathematical model of material behavior is that it provides a useful description of the salient features of the behavior of a real material *under limited environmental conditions*. Under different conditions of operation a structural metal may behave as though it were elastic, viscoelastic, elastic–plastic, or fluid. Consequently, we limit consideration here to conditions encountered in the penetration problem, for example, where metals are subjected to high-speed loading resulting in a multi-dimensional stress state with a large (up to 100 kb, say) hydrostatic component. A thermoelastic effect is thus dominant, that is, the main resistance to deformation is due to compressibility, and material rigidity offers only secondary resistance. Since plasticity effects are only secondary, we employ a very simple description of plastic flow incorporating such simplifications as rate-independence and isotropy. Our objective is not to classify material behavior through the formulation of very general and complex constitutive equations, but rather to suggest relations which might be practically used in the numerical solution of boundary value problems.

Fundamental to the development presented here is the recent work of Lee and his colleagues. In [1] and [2] the groundwork is presented for the analysis of plane waves in

elastic–plastic materials at finite strain and in [3] approximate constitutive equations are proposed. In [4] Lee generalizes some of the previous work to three-dimensional stress states. Very general discussions of constitutive laws have been presented by Green and Naghdi [5], Perzyna [6] and Sedov [7], but these aim more at material classification than at application. The kinematics employed is based on the works of Lee *et al.*, Freund [8] and the monograph of Truesdell and Toupin [9], and we include only enough results to establish the notation.

Most mathematical symbols and equations in this section, as well as in following sections, will be written in direct (matrix) notation, because this form seems to lend itself to quicker and easier interpretation. In special instances, however, we will resort to tensor (index) notation to emphasize important results or to exhibit particular detail. When the tensor notation is employed, upper case Latin indices relate to the initial configuration, lower case Greek to the reference configuration, and lower case Latin to the current configuration.

As in [1–4], the deformation at any time is viewed as a rigid–plastic deformation of the initial configuration into an imaginary, intermediate configuration superimposed on a thermoelastic deformation of the intermediate configuration into the current configuration. With the introduction of this intermediate state, it can be shown that the total rate of deformation can be written as the sum of appropriately defined rates of elastic and plastic deformation. The total deformation is described by the deformation gradient $\mathbf{F}$, which is defined by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}, \tag{1}$$

$$\mathbf{x} = \mathbf{x}(X_0, t) \tag{2}$$

where $\mathbf{x}$ is the position at time t of the particle originally at $\mathbf{X}_0$. With the introduction of the intermediate reference configuration, $\mathbf{F}$ has the noncommutative representation

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \tag{3}$$

where $\mathbf{F}^e$ and $\mathbf{F}^p$ are the tensors defining elastic and plastic deformation, respectively. In general, $\mathbf{F}^e$ and $\mathbf{F}^p$ are not gradients of vectors. As pointed out in [4], the decomposition (3) is not unique because the reference configuration is not unique. The most convenient strain measure for discussing the elastic deformation is $\mathbf{C}$ defined by

$$\mathbf{C} = (\mathbf{F}^e)^T\mathbf{F}^e. \tag{4}$$

This is the familiar Cauchy–Green tensor of the theory of elasticity. The tensor $\mathbf{C}$ is related to the material elastic strain $\mathbf{E}^e$ by

$$2\mathbf{E}^e = \mathbf{C} - \mathbf{I}. \tag{5}$$

The fundamental measure of rate of change of configuration will be taken to be the stretching tensor, which defines a measure of the rate of change of current length per unit current length. This measure depends only on the current configuration. The stretching tensor $\mathbf{D}$ is defined by

$$\mathbf{D} = \text{sym}(\dot{\mathbf{F}}\mathbf{F}^{-1}). \tag{6}$$

It is shown in [8] that, even though $\dot{\mathbf{F}}^e$ and $\dot{\mathbf{F}}^p$ are not velocity gradients, $\mathbf{D}$ may be written as the sum of an elastic stretching $\mathbf{D}^e$ and a plastic stretching $\mathbf{D}^p$, where

$$\mathbf{D}^e = \text{sym}(\dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}}), \tag{7}$$

$$\mathbf{D}^p = \text{sym}(\mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}\mathbf{F}^{e^{-1}}), \tag{8}$$

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p. \tag{9}$$

The elastic stretching is the rate of deformation of the current state with respect to the reference state, while the plastic stretching is the rate of deformation of the current configuration due to the changing reference configuration. The elastic stretching is related to $\mathbf{C}$ by

$$\mathbf{C} = 2(\mathbf{F}^e)^T\mathbf{D}^e\mathbf{F}^e. \tag{10}$$

The antisymmetric tensors corresponding to the stretchings are the spins, denoted by $\mathbf{W}$, $\mathbf{W}^e$, $\mathbf{W}^p$. The components of the spin tensors are, essentially, the averages of the angular velocities of line elements along two coordinate directions about the third coordinate direction. Furthermore, $\mathbf{W} = \mathbf{W}^e + \mathbf{W}^p$.

One additional quantity of interest is the stretching of the reference configuration, defined by

$$\mathbf{D}^s = \text{sym}(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}), \tag{11}$$

the corresponding spin being $\mathbf{W}^s$. Following [9], $\mathbf{D}^s$ is called the slippage rate period. The corresponding spins are related to the plastic stretching and spin by

$$\mathbf{F}^e(\mathbf{D}^s + \mathbf{W}^s)\mathbf{F}^{e^{-1}} = \mathbf{D}^p + \mathbf{W}^p, \tag{12}$$

$$\mathbf{F}^{e^{-1^T}}(\mathbf{D}^s - \mathbf{W}^s)\mathbf{F}^{e^T} = \mathbf{D}^p - \mathbf{W}^p. \tag{13}$$

THERMODYNAMICS

The formulation of a problem in continuum mechanics which involves the coupling of mechanical and thermal effects must be based on a thermodynamical theory. Unfortunately, the development of a thermodynamical theory of plasticity which is general enough to cover a reasonable range of material behavior, yet specific enough to be useful, is in a preliminary stage. The problem is being attacked, on the one side, by way of a general, mathematically consistent phenomenological approach, e.g. [6], and on the other by way of a detailed microstructural approach, e.g. [10]. In view of the situation, we fulfill our immediate needs by adopting the simple phenomenological theory of Lee [4]. Since the intended application is to metals at high deformation rates, we assume the deformation process to be adiabatic.

Lee's thermodynamic description is based on a weak interaction of a thermoelastic system and a simplified plastic system. It is assumed that, at any instant, some fraction, say γ, of the plastic work is converted into heat, the remainder of the plastic work being stored as a residual free energy due to elastic distortion of the crystal lattice in the neighborhood of dislocations. It is assumed, further, that this residual free energy does not affect the stress distribution. The heat generated by plastic work is taken to be an internal heat

source for the thermoelastic system. These assumptions are incorporated in writing the laws of thermodynamics in the following form:

$$\rho\dot{\psi}_{\text{res}} = (1-\gamma)\,\text{tr}(\mathbf{TD}^p) \tag{14a}$$

$$\rho(\dot{\psi}+s\dot{\theta}+\dot{s}\theta) = \text{tr}(\mathbf{TD}^e)+\gamma\,\text{tr}(\mathbf{TD}^p) \tag{14b}$$

$$\dot{s}-\gamma\,\text{tr}(\mathbf{TD}^p)/\theta \geq 0 \tag{14c}$$

where ρ is current mass density, ψ_{res} is residual specific free energy, $\mathbf{T}$ is the true stress, ψ is elastic specific free energy, s is specific entropy and θ is absolute temperature. Equations (14a, b) together are a statement of the first law of thermodynamics, (14c) being a statement of the second law.

Before proceeding, the various descriptions of stress employed are defined. The tensor $\mathbf{T}$, already introduced in (14), represents the true stress. A second stress tensor is defined in terms of the true stress and the elastic deformation by

$$\mathbf{T}^l/\rho^r = (\mathbf{T}/\rho)(\mathbf{F}^{e^{-1}})^T, \tag{15a}$$

and is referred to as the first Piola–Kirchhoff stress tensor, the Lagrange stress tensor, or the mixed stress tensor. In (15a), ρ^r is the density of the reference configuration. Another useful description of stress is defined by

$$\mathbf{T}^k/\rho^r = \mathbf{F}^{e^{-1}}(\mathbf{T}/\rho)(\mathbf{F}^{e^{-1}})^T. \tag{15b}$$

The tensor $\mathbf{T}^k$ is referred to as the second Piola–Kirchhoff stress tensor, the Kirchhoff stress tensor, or the material stress tensor.

Since it is desired to write the rate of deformation and entropy in terms of stress and temperature, the appropriate thermodynamic potential is the specific free enthalpy χ, which is related to the specific free energy by

$$\chi = \psi - \text{tr}[(\mathbf{T}^l/\rho^r)(\mathbf{F}^e)^T]. \tag{16}$$

Substituting (16) and (14b) into (14c) and employing the fact that, at any given stress and temperature, the stress rate and temperature rate are arbitrary, we obtain the result

$$\mathbf{F}^e = \frac{\partial\chi}{\partial(\mathbf{T}^l/\rho^r)}, \qquad \dot{s} = -\frac{\partial\chi}{\partial\theta}, \tag{17a}$$

$$\rho\theta\dot{s} = \gamma\,\text{tr}[\mathbf{TD}^p]. \tag{17b}$$

Relations (17a) are the constitutive equations for the thermoelastic process, and (17b) couples the thermoelastic and plastic processes. We cannot derive useful restrictions on the constitutive equations for plasticity from (14a) similar to (17a), simply because we do not yet know how to select a reasonable and consistent set of state variables and state functionals [11].

In general, at each material point, the quantity γ will depend on the thermomechanical history. However, the variation of γ during any actual process is usually small (see [1]), and the value of γ is assumed to be constant and in the range $0{\cdot}9 \leq \gamma \leq 1{\cdot}0$.

GENERAL FORM OF THE CONSTITUTIVE EQUATION

The weak coupling of elastic and plastic effects discussed in relation to the thermodynamics allows us to consider the salient elastic and plastic features of the deformation process separately. The kinematics of the problem suggests that a reasonable way of combining the separate results is to seek the elastic stretching and plastic stretching in terms of stress and temperature and then add the two rates as indicated in [8]. There is also a geometric coupling between elastic and plastic effects due to the fact that the reference state of elastic deformation varies with plastic flow, and also that the description of plastic deformation is given in the reference configuration, since this is the configuration of the body due to plastic deformation alone.

Relation (17a) gives the general form of the elastic constitutive equation for a material described by a free enthalpy function. Assuming the enthalpy to be a function of Kirchhoff stress and temperature, application of the chain rule to (17a) yields the equivalent, but more convenient, form

$$\mathbf{C} = \frac{\partial \chi}{\partial(\mathbf{T}^k/\rho^r)} \equiv \Sigma(\mathbf{T}^k/\rho^r, \theta). \tag{18}$$

Suppose that the material under consideration is elastically isotropic. In the usual theory of elasticity, the reference or undistorted configuration and the initial configuration coincide. Material isotropy is then equivalent to the condition that the response of the material to any stress state, measured from the reference configuration, is invariant under rotation of the reference configuration. If the constitutive equation is written with respect to a basis fixed in the undistorted reference configuration, i.e. a material basis, then material isotropy is expressed by the condition that the response function be an isotropic tensor-valued function of its arguments. The situation is somewhat different here. The reference configuration, which varies with time, is distinct from the fixed initial configuration. At each instant of time, however, a constitutive equation such as (18) is written with respect to a basis in the reference configuration, which is not a material basis. Consequently, the condition on the response function imposed by the assumption of isotropy must be modified.

This modification proceeds as follows. Limiting consideration to a generic material particle and any instant of time, we first transform the constitutive equation (18) from the fixed spatial basis to any orthogonal basis fixed in the material. This transformation is defined by the orthogonal tensor $\mathbf{Q}_1$. At the same material particle but at a different instant of time, the material lines which had coincided with the above material basis are no longer orthogonal and a different material basis must be selected. That is, at each instant of time, $\mathbf{Q}_1$ is a transformation from the fixed spatial basis to an orthogonal material basis, but the material base vectors coincide with different material line elements at different times, and $\mathbf{Q}_1 = \mathbf{Q}_1(t)$. While it does not play a major role in the discussion of isotropy, the way in which $\mathbf{Q}_1$ varies with time is very important in the subsequent discussion of rate equations. The constitutive equation resulting from this transformation is a description of material behavior with respect to a material basis in the reference configuration, and the isotropy condition may then be imposed in the usual way by introducing the constant orthogonal tensor $\mathbf{Q}_2$. Setting $\mathbf{Q} = \mathbf{Q}_2\mathbf{Q}_1$, the assumption of elastic isotropy implies that the response function $\boldsymbol{\Sigma}$ must satisfy the condition

$$\mathbf{Q}\boldsymbol{\Sigma}(\mathbf{T}^k/\rho^r, \theta)\mathbf{Q}^T = \boldsymbol{\Sigma}(\mathbf{Q}\mathbf{T}^k\mathbf{Q}^T/\rho^r, \theta) \tag{19}$$

for all $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Relation (19) does not differ in general form from the equivalent condition

of the usual theory of elasticity. As we have discussed, however, $\mathbf{Q}$ has a very special interpretation in our case.

The desired rate law of elasticity is obtained by differentiating (19) with respect to time at each material point. We encounter the usual kinematical difficulty that, even though certain quantities transform as tensors under general observer transformations, the material time derivatives of these quantities do not transform as tensors. By observer frame, or simply observer, we mean a basis undergoing some motion, and an observer transformation defines the relative motion of two observers. If two observers are not undergoing relative motion they are said to be equivalent. Since any constitutive equation must be independent of the observer, the elasticity rate law must be written in terms of frame-indifferent stress rates. Such rates are introduced in the following manner [9]. First, we choose a particular, or preferred, observer frame which is selected because the time derivative of the stress tensor transforms as a tensor under the class of transformations among all frames equivalent to the preferred frame. A general stress rate is then sought which (a) reduces to the time derivative of stress in the preferred frame, and (b) transforms as a tensor under the class of all observer transformations. Since $\mathbf{Q}_1$ is a transformation to a material basis, its presence in (19) suggests an observer rotating with the material as the preferred observer, and this is the point of view we pursue.

In the usual terminology, an observer frame which is instantaneously fixed in the body is called a co-rotational frame. At any material point and any time, there are an infinite number of co-rotational frames, all of which are equivalent observer frames. Because of this equivalence, we may choose one co-rotational frame over another, for the sake of convenience, without losing generality of the kinematic description. The most convenient choice is that co-rotational frame which instantaneously coincides with the fixed spatial basis at that point, that is, $\mathbf{Q} = \mathbf{I}$. However, since the co-rotational frame is instantaneously material, $\dot{\mathbf{Q}} = \mathbf{W}^s$. Carrying out the calculations suggested by conditions (a) and (b), with co-rotational frame as the preferred frame, we are led to the following definition of a frame-indifferent rate for any symmetric second order tensor $\mathbf{A}$,

$$\mathring{\mathbf{A}} = \dot{\mathbf{A}} - \mathbf{W}^s\mathbf{A} + \mathbf{A}\mathbf{W}^s. \tag{20a}$$

This rate measure is called the co-rotational rate (with respect to the reference configuration). As pointed out by Prager [12], it has the property that

$$\overset{\circ}{\overline{\mathbf{A}_1\mathbf{A}_2}} = \mathring{\mathbf{A}}_1\mathbf{A}_2 + \mathbf{A}_1\mathring{\mathbf{A}}_2. \tag{20b}$$

Relation (19) is now differentiated with respect to time for arbitrary $\mathbf{Q}$, keeping in mind that $\mathbf{Q}$ is a transformation from the fixed spatial basis to any material basis. After differentiation the choice of preferred frame is made, i.e. we set $\mathbf{Q} = \mathbf{I}$ and $\dot{\mathbf{Q}} = \mathbf{W}^s$, and the following relation results†

$$\dot{\mathbf{C}} = \underset{\sim}{\mathbf{K}}^e\overset{\circ}{[\overline{\mathbf{T}^k/\rho^r}]} + \mathbf{M}^e\dot{\theta} + \mathbf{W}^s\mathbf{C} - \mathbf{C}\mathbf{W}^s, \tag{21a}$$

where

$$\underset{\sim}{\mathbf{K}}^e = \partial\boldsymbol{\Sigma}/\partial(\mathbf{T}^k/\rho^r), \qquad \mathbf{M}^e = \partial\boldsymbol{\Sigma}/\partial\theta. \tag{21b}$$

A similar calculation is carried out by Noll in Section 15 of [13].

† Fourth order tensors, which are viewed as linear operators in the nine-dimensional space of second order tensors, are denoted by boldface with a wavy underscore, $\sim$, and the argument of the operator is put in square brackets. In index notation $(\underset{\sim}{\mathbf{K}}[\mathbf{T}])_{ij} = K_{ijkl}T^{kl}$.

The final form of the constitutive law which we seek is an expression for the total stretching in terms of true stress and temperature. We hope to reach this form by expressing the elastic stretching and the plastic stretching, separately, in terms of true stress and temperature, and then taking advantage of the additivity of the stretching. To transform (21) into the desired form, consider first the co-rotational stress rate. From (15b) we have

$$\overset{\circ}{(\mathbf{T}^k/\rho^r)} = \overset{\circ}{[\mathbf{F}^{e^{-1}}(\mathbf{T}/\rho)(\mathbf{F}^{e^{-1}})^T]}. \tag{22}$$

Decomposing $\mathbf{F}^e$ into a pure rotation $\mathbf{R}^e$ followed by a pure deformation $\mathbf{V}^e$ as

$$\mathbf{F}^e = \mathbf{V}^e\mathbf{R}^e, \tag{23}$$

we carry out the operation indicated on the right side of (22). Using the kinematic relation

$$\dot{\mathbf{R}}^e = \mathbf{W}\mathbf{R}^e - \mathbf{R}^e\mathbf{W}^s, \tag{24}$$

we arrive at the result

$$\overset{\circ}{(\mathbf{T}^k/\rho^r)} = (\mathbf{R}^e)^T \overset{\nabla}{\overline{\mathbf{V}^{e^{-1}}(\mathbf{T}/\rho)\mathbf{V}^{e^{-1}}}}\mathbf{R}^e. \tag{25}$$

The frame-indifferent rate introduced on the right side of (25) is the co-rotational rate taken with respect to the current configuration and is defined, for any symmetric tensor $\mathbf{A}$, by

$$\overset{\nabla}{\mathbf{A}} = \dot{\mathbf{A}} - \mathbf{W}\mathbf{A} + \mathbf{A}\mathbf{W}. \tag{26}$$

An expression for the elastic stretching in terms of $\dot{\mathbf{C}}$ is obtained from (11) as

$$\begin{aligned} 2\mathbf{D}^e &= (\mathbf{F}^{e^{-1}})^T\dot{\mathbf{C}}\mathbf{F}^{e^{-1}} \\ &= \mathbf{V}^{e^{-1}}\mathbf{R}^e\dot{\mathbf{C}}(\mathbf{R}^e)^T\mathbf{V}^{e^{-1}}. \end{aligned} \tag{27}$$

Consequently, the constitutive equation for the elastic part of the deformation is

$$2\mathbf{D}^e = \mathbf{V}^{e^{-1}}\mathbf{G}^e\overset{\nabla}{\overline{[\mathbf{V}^{e^{-1}}(\mathbf{T}/\rho)\mathbf{V}^{e^{-1}}]}}\mathbf{V}^{e^{-1}} + \mathbf{V}^{e^{-1}}\mathbf{H}^e\dot{\theta}\mathbf{V}^{e^{-1}} + (\mathbf{F}^{e^{-1}})^T\mathbf{W}^s(\mathbf{F}^e)^T - \mathbf{F}^e\mathbf{W}^s\mathbf{F}^{e^{-1}} \tag{28}$$

where

$$\begin{aligned} \mathbf{G}^e &= \mathbf{R}^e\{\mathbf{R}^e[(\mathbf{K}^e\mathbf{R}^{e^T})\mathbf{R}^{e^T}]\} \\ \mathbf{H}^e &= \mathbf{R}^e\mathbf{M}^e\mathbf{R}^{e^T}. \end{aligned} \tag{29}$$

In index notation, omitting the superscript e,

$$\begin{aligned} G_{ijkl} &= R_{i\beta}R_{j\alpha}K^{\alpha\beta\mu\gamma}R_{k\gamma}R_{l\mu} \\ H_{ij} &= R_{i\alpha}M^{\alpha\beta}R_{j\beta}. \end{aligned} \tag{29a}$$

Consider now the irreversible or plastic part of the deformation process. As stated above, the reference, or intermediate, configuration is the state the body would be in due to plastic deformation alone. We take the point of view that this configuration is the current state of a plastic–rigid material and discuss the changing reference state accordingly. The variables used to describe the material behavior will all be defined in the reference state. The concept of finite strain is not used here, implying that the plastic–rigid material does not have a "natural" state. Furthermore, for the applications we have in mind, the

deformation process is taken to be rate-independent. That is, the instantaneous state of the body depends on all previous states only through the order in which they have occurred. The apparent stress in the reference state is called $\mathbf{T}^r$, and is defined for elastically isotropic materials by

$$\frac{1}{\rho^r}\,\mathrm{tr}[\mathbf{T}^r\mathbf{D}^s] = \frac{1}{\rho}\,\mathrm{tr}[\mathbf{T}\mathbf{D}^p], \tag{30}$$

where ρ^r and ρ are mass densities in the reference and current configurations. Relation (30) states that the rate of plastic work per unit mass in the reference configuration is the actual observed rate of dissipation per unit mass, independent of the elastic deformation. Assuming (30) to hold for arbitrary deformation processes, we can solve for $\mathbf{T}^r$ in terms of $\mathbf{T}$. Because the trace operator is linear, and because the spin tensors are antisymmetric, (30) can be rewritten

$$\frac{1}{\rho^r}\,\mathrm{tr}[\mathbf{T}^r(\mathbf{D}^s+\mathbf{W}^s)] = \frac{1}{\rho}\,\mathrm{tr}[\mathbf{T}(\mathbf{D}^p+\mathbf{W}^p)]. \tag{30'}$$

Employing (13), this relation becomes

$$\mathrm{tr}\left[\left(\frac{1}{\rho^r}\mathbf{T}^r - \frac{1}{\rho}\mathbf{F}^{e^{-1}}\mathbf{T}\mathbf{F}^e\right)(\mathbf{D}^s+\mathbf{W}^s)\right] = 0. \tag{31}$$

If this must hold for all possible deformations, then we conclude that

$$\mathbf{T}^r/\rho^r = \mathbf{F}^{e^{-1}}\mathbf{T}\mathbf{F}^e/\rho. \tag{32}$$

As in (23), $\mathbf{F}^e$ has the representation

$$\mathbf{F}^e = \mathbf{V}^e\mathbf{R}^e,$$

where the symmetric tensor $\mathbf{V}^e$ is a pure deformation and $\mathbf{R}^e$ is an orthogonal tensor. For the elastically isotropic materials being considered, the principal directions of $\mathbf{T}$ and $\mathbf{V}^e$ coincide and, therefore, $\mathbf{T}$ and $\mathbf{V}^e$ commute under multiplication. Relation (31) may be written in the form

$$\mathbf{T}^r/\rho^r = (\mathbf{R}^e)^T\mathbf{T}\mathbf{R}^e/\rho. \tag{33}$$

The stress $\mathbf{T}^r$ is not the Kirchhoff stress, but is instead an apparent stress producing plastic flow of the reference configuration. The apparent stress is related to the actual stress through the work equality (30).

We assume the law governing the plastic deformation to be of the rate type. In particular, we assume that the instantaneous rate of deformation of the reference configuration is determined uniquely by the instantaneous apparent stress and stress rate, and a set of scalar functions of position and time α_i, $i = 1, 2, \ldots, n$, and their derivatives. Using the preceding statement implies the existence of a function $\mathbf{\Delta}$ such that

$$\mathbf{L}^s = \mathbf{\Delta}(\mathbf{T}^r/\rho^r, \overset{\cdot}{\mathbf{T}^r/\rho^r}, \alpha_i, \dot{\alpha}_i). \tag{34}$$

Because $\mathbf{L}^s$ and $\mathbf{T}^r$ transform as tensors only under the class of transformations among frames rotating with the material, we must interpret (34) as a tensor equation in an arbitrary

frame rotating with the material. Suppose the transformation from the fixed spatial frame to this rotating frame is defined by $\mathbf{R}$. Relation (34) may then be rewritten as

$$\mathbf{R}\mathbf{D}^s\mathbf{R}^T = \boldsymbol{\Delta}(\mathbf{R}\mathbf{T}^r\mathbf{R}^T/\rho^r, \mathbf{R}\overset{\circ}{\overline{(\mathbf{T}^r/\rho^r)}}\mathbf{R}^T, \alpha_i, \dot{\alpha}_i). \tag{35}$$

In particular, at each instant of time we can select that co-rotational frame which coincides with the spatial frame, that is, $\mathbf{R} = \mathbf{I}$, so that

$$\mathbf{D}^s = \boldsymbol{\Delta}(\mathbf{T}^r/\rho^r, \overset{\circ}{\overline{\mathbf{T}^r/\rho^r}}, \alpha_i, \dot{\alpha}_i). \tag{36}$$

Combining (35) and (36), we obtain the condition on $\boldsymbol{\Delta}$

$$\mathbf{R}\boldsymbol{\Delta}(\mathbf{T}^r/\rho^r, \overset{\circ}{\overline{\mathbf{T}^r/\rho^r}}, \alpha_i, \dot{\alpha}_i)\mathbf{R}^T = \boldsymbol{\Delta}(\mathbf{R}\mathbf{T}^r\mathbf{R}^T/\rho^r, \mathbf{R}\overset{\circ}{\overline{(\mathbf{T}^r/\rho^r)}}\mathbf{R}^T, \alpha_i, \dot{\alpha}_i). \tag{37}$$

Because $\mathbf{R}$ is arbitrary in (35), condition (37) states that $\boldsymbol{\Delta}$ is an isotropic function of its arguments and, consequently, (36) is an isotropic flow rule. The assumption of rate independence is incorporated by writing (36) in the form

$$\mathbf{D}^s = \mathbf{K}^s\left[\overset{\circ}{\overline{(\mathbf{T}^r/\rho^r)}} + \sum_i \mathbf{M}_i\dot{\alpha}_i\right], \tag{38}$$

where the fourth order tensor $\underset{\sim}{\mathbf{K}}^s$ and the set of second order tensors $\mathbf{M}_i$, $i = 1, 2, \ldots, n$, are functions of $\mathbf{T}^r/\rho^r$ and α_i. The response functions $\underset{\sim}{\mathbf{K}}^s$ and $\mathbf{M}_i$ could also depend on ratios of the components of $\mathbf{T}^r$ and/or $\dot{\alpha}_i$, and still represent a rate independent process. The point $\overset{\circ}{\mathbf{T}}{}^r = 0$, $\dot{\alpha}_i = 0$ would then be a branch point of the response functions, however, and we preclude such a situation by writing (38). It can be shown for the particular case of isothermal deformation of rate independent stable materials that $\underset{\sim}{\mathbf{K}}^s$ has no such branch points [14].

We can draw several possibilities for the set of scalar functions α_i from the theory of infinitesimal plastic deformation [15]. The temperature is, of course, included in the set. Other frequently selected parameters are the arclength of plastic strain history in nine-dimensional strain space and the accumulated plastic work per unit mass. Consequently, we might set

$$\begin{aligned} \alpha_1 &= \theta, \qquad \alpha_2 = \int_0^t \frac{1}{\rho^r}\,\mathrm{tr}[\mathbf{T}^r\mathbf{D}^s]\,\mathrm{d}\tau, \\ \alpha_3 &= \int_0^t \{\mathrm{tr}[((\mathbf{F}^p)^T\mathbf{D}^s\mathbf{F}^p)^2]\}^{\frac{1}{2}}\,\mathrm{d}\tau, \end{aligned} \tag{39a}$$

where we tacitly assume that the process begins at $t = 0$. A further possibility is to allow one or more of the scalar functions to weight the history of another. For example, Lee [4] employs

$$\alpha_4 = \int_0^t \frac{h(\alpha_1)}{\rho^r}\,\mathrm{tr}[\mathbf{T}^r\mathbf{D}^s]\,\mathrm{d}\tau. \tag{39b}$$

The quantity α_4 is considered to be a hardening parameter and h is an increasing function of α_1. Monotonicity of h is deduced from the argument that by increasing the temperature of a metal we can achieve a certain dislocation density, or degree of hardness, through the expenditure of a decreasing amount of plastic work.

We can generalize the notion of "effective" stress, discussed by Prager [16], if the constitutive equation depends on the set α_i only through a single scalar function $\sigma(\alpha_i)$ which modulates the stress in some manner. The existence of such a σ implies the representation of $\mathbf{M}_i$ in (38)

$$\mathbf{M}_i = \frac{\partial \sigma}{\partial \alpha_i} \mathbf{T}^r/\rho^r \sigma,$$

and the flow law takes the form

$$\mathbf{D}^s = \sigma^{-1} \underset{\sim}{\mathbf{K}}^s \overset{\circ}{\overline{[\sigma \mathbf{T}^r/\rho^r]}}, \tag{40}$$

where $\underset{\sim}{\mathbf{K}}^s$ depends only on $\sigma \mathbf{T}^r/\rho^r$. The scalar stress tensor $\sigma \mathbf{T}^r$ is called the effective stress.

If we let $\mathbf{B}$ be the "argument" of $\mathbf{K}^s$ in either (38) or (40), then the most general form of $\mathbf{K}^s[\mathbf{B}]$ is [12]

$$\begin{aligned}
\mathbf{K}^s[\mathbf{B}] &= \{\Lambda_1 \operatorname{tr}(\mathbf{B}) + \Lambda_2 \operatorname{tr}(\mathbf{T}^r\mathbf{B})/\rho^r + \Lambda_3 \operatorname{tr}(\mathbf{T}^{r2}\mathbf{B})/(\rho^r)^2\}\mathbf{I} \\
&+ \{\Lambda_4 \operatorname{tr}(\mathbf{B}) + \Lambda_5 \operatorname{tr}(\mathbf{T}^r\mathbf{B})/\rho^r + \Lambda_6 \operatorname{tr}(\mathbf{T}^{r2}\mathbf{B})/(\rho^r)^2\}\mathbf{T}^r/\rho^r \\
&+ \{\Lambda_7 \operatorname{tr}(\mathbf{B}) + \Lambda_8 \operatorname{tr}(\mathbf{T}^r\mathbf{B})/\rho^r + \Lambda_9 \operatorname{tr}(\mathbf{T}^{r2}\mathbf{B})/(\rho^r)^2\}(\mathbf{T}^r/\rho^r)^2 \\
&+ \Lambda_{10}\mathbf{B} + \Lambda_{11}(\mathbf{B}\mathbf{T}^r + \mathbf{T}^r\mathbf{B})/\rho^r + \Lambda_{12}(\mathbf{B}\mathbf{T}^{r2} + \mathbf{T}^{r2}\mathbf{B})/(\rho^r)^2,
\end{aligned} \tag{41}$$

where the twelve scalars Λ_k, $k = 1, \ldots, 12$ are functions of the α_i and the invariants of $\mathbf{T}^r/\rho^r$ for the form (38), or of the invariants of $\sigma \mathbf{T}^r/\rho^r$ for the form (40).

In addition to the flow rule for the rigid–plastic material, we prescribe a yield function by which we determine whether the material is indeed flowing under a given stress state and set α_i, or whether it remains rigid. Thus, it is supposed that, in the nine-dimensional space of second order tensors, a surface exists and is defined by the equation

$$f(\mathbf{T}^r/\rho^r, \alpha^i) = 0. \tag{42}$$

The surface is defined so that for $f < 0$ the material remains rigid, while for $f = 0$ flow takes place; the region $f > 0$ is inaccessible. For $f < 0$, $\underset{\sim}{\mathbf{K}}^s \equiv 0$; for $f = 0$, $\underset{\sim}{\mathbf{K}}^s$ has the form given in (41).

Whereas (38) expresses $\mathbf{D}^s$ in terms of apparent stress and α_i, it is a representation of $\mathbf{D}^p$ in terms of true stress and α_i that we require. To express $\mathbf{T}^r/\rho^r$ in terms of the true stress, we apply the definition (20) to (33)

$$\begin{aligned}
\overset{\circ}{\overline{\mathbf{T}^r/\rho^r}} &= (\dot{\mathbf{R}}^e)^T\mathbf{T}\mathbf{R}^e/\rho + (\mathbf{R}^e)^T\overset{\circ}{\overline{(\mathbf{T}/\rho)}}\mathbf{R}^e + (\mathbf{R}^e)^T\mathbf{T}\dot{\mathbf{R}}^e/\rho \\
&\quad - \mathbf{W}^s(\mathbf{R}^e)^T\mathbf{T}\mathbf{R}^e/\rho + (\mathbf{R}^e)^T\mathbf{T}\mathbf{R}^e\mathbf{W}^s/\rho.
\end{aligned} \tag{43}$$

Using the kinematic relation (24), equation (43) reduces to

$$\overset{\circ}{\overline{(\mathbf{T}^r/\rho^r)}} = (\mathbf{R}^e)^T\overset{\nabla}{\overline{(\mathbf{T}/\rho)}}\mathbf{R}^e, \tag{44}$$

where the definition (26) has been employed on the right side of (44). Adding the kinematic relations (12) and (13), decomposing $\mathbf{F}^e$ according to (23), and substituting the representation of $\mathbf{D}^s$ given in (38), we obtain

$$2\mathbf{D}^p = \mathbf{V}^e\underset{\sim}{\mathbf{G}}^p[\overset{\nabla}{\overline{\mathbf{T}/\rho}}+\mathbf{N}_i\dot{\alpha}_i]\mathbf{V}^{e^{-1}}+\mathbf{V}^{e^{-1}}\underset{\sim}{\mathbf{G}}^p[\overline{\mathbf{T}/\rho}+\mathbf{N}_i\dot{\alpha}_i]\mathbf{V}^e+\mathbf{F}^e\mathbf{W}^s\mathbf{F}^{e^{-1}}-(\mathbf{F}^{e^{-1}})^T\mathbf{W}^s(\mathbf{F}^e)^T. \tag{45}$$

In (45),

$$\begin{aligned}\underset{\sim}{\mathbf{G}}^p &= \mathbf{R}^e\{\mathbf{R}^e[(\underset{\sim}{\mathbf{K}}^s\mathbf{R}^{e^T})\mathbf{R}^{e^T}]\},\\ \mathbf{N}^i &= \mathbf{R}^e\mathbf{M}^i\mathbf{R}^{e^T}.\end{aligned} \tag{46}$$

Finally, by adding (28) and (45), we get the general form of the constitutive law

$$\begin{aligned}\mathbf{D} = \tfrac{1}{2}\mathbf{V}^{e^{-1}}\{&\underset{\sim}{\mathbf{G}}^e[\overset{\nabla}{\overline{\mathbf{V}^{e^{-1}}(\mathbf{T}/\rho)\mathbf{V}^{e^{-1}}}}]+\mathbf{H}^e\dot{\theta}+\mathbf{V}^{e^2}\underset{\sim}{\mathbf{G}}^p[\overset{\nabla}{\overline{\mathbf{T}/\rho}}+\mathbf{N}_i\dot{\alpha}_i]\\ &+\underset{\sim}{\mathbf{G}}^p[\overset{\nabla}{\overline{\mathbf{T}/\rho}}+\mathbf{N}_i\dot{\alpha}_i]\mathbf{V}^{e^2}\}\mathbf{V}^{e^{-1}}.\end{aligned} \tag{47}$$

Observe that the terms depending on spin $\mathbf{W}^s$ have cancelled. The main physical assumptions incorporated in (47) are elastic isotropy and rate-independence. In the following section, the consequences of several more assumptions, valid for most metals undergoing finite deformation, will be investigated.

RESULT OF PHYSICAL ASSUMPTIONS

What we seek is a description of the behavior of most metals subjected to the particular environmental conditions of large stress, high speed loading. Under the action of the large pressures encountered in these situations, most metals can sustain large but recoverable volume changes. However, during elastic–plastic deformation of most metals, yielding begins before elastic shear strains increase to the point where they must be considered large. Thus we assume that the elastic shear strain is small, even though the plastic shears may be large.

The above physical assumptions are incorporated, albeit indirectly, by introducing a characteristic material constant with the dimensions of stress, say K, and writing

$$\begin{aligned}\mathbf{T}^k/K &= \mathbf{S}^k/K-\mathbf{I}P^k/K,\\ P^k &= -\tfrac{1}{3}\,\mathrm{tr}[\mathbf{T}^k].\end{aligned} \tag{48}$$

It is then assumed that the tensor $\mathbf{S}^k/K$ is small, that is, the magnitude of each element of the tensor is much less than unity. A reasonable choice for K would be the initial slope of a pressure vs. dilatational strain curve, which is the bulk modulus of linear elasticity theory. For example, suppose we want to describe the behavior of steel ($K \simeq 3\times 10^7$ psi) under environmental conditions involving pressures of up to $1{\cdot}5\times 10^6$ psi. (For higher pressures a hydrodynamic theory is usually applicable.) Then $P^k/K \leq 0{\cdot}05$, and a power series representation of a deformation measure about $P^k/K = 0$ should converge quickly.

The goal of this section is to obtain an expression for the elastic response function $\partial\mathbf{C}/\partial(\mathbf{T}^k/\rho^r)$. The calculations which must be carried out are simplified somewhat if we

assume here that the net volume change associated with plastic deformation is zero which implies that $\rho^r = \rho_0$, a constant. Relation (21b) becomes

$$\underset{\sim}{\mathbf{K}}^e = \rho_0 \frac{\partial \mathbf{C}}{\partial \mathbf{T}^k}. \tag{49}$$

Whereas the assumption concerning plastic volume change would topically be more appropriate in the subsequent discussion of the plasticity law, it is convenient to employ it here.

For an isotropic elastic material the free enthalpy is an invariant and, therefore,

$$\chi = \chi(\mathrm{I}^k, \mathrm{II}^k, \mathrm{III}^k)$$

where the arguments of χ are the three principal invariants of $\mathbf{T}^k$. Carrying out the differentiation process indicated in (49) we obtain

$$\mathbf{C} = \rho_0\{A_0\mathbf{I} + A_1\mathbf{T}^k/K + A_2(\mathbf{T}^k/K)^2\}, \tag{50}$$

where

$$A_0 = \left(\frac{\partial \chi}{\partial \mathrm{I}^k} - \mathrm{I}^k\frac{\partial \chi}{\partial \mathrm{II}^k} + \mathrm{II}^k\frac{\partial \chi}{\partial \mathrm{III}^k}\right),$$

$$A_1 = K\left(-\frac{\partial \chi}{\partial \mathrm{II}^k} - \mathrm{I}^k\frac{\partial \chi}{\partial \mathrm{III}^k}\right), \qquad A_2 = K^2\frac{\partial \chi}{\partial \mathrm{III}^k}. \tag{51}$$

The assumed form of the free enthalpy function is a power series expansion in the invariants I^k, II^k, III^k. Introducing the dimensionless sets of temperature dependent material parameters a_i, b_i, c_i, d_i, we write

$$\chi = \frac{K}{\rho_0}\{a_0 + a_1\mathrm{I}^k/K + b_1(\mathrm{I}^k/K)^2 + b_2\mathrm{II}^k/K^2 + c_1(\mathrm{I}^k/K)^3 + c_2\mathrm{I}^k\mathrm{II}^k/K^3 + c_3\mathrm{III}^k/K^3$$

$$+ d_1(\mathrm{I}^k/K)^4 + d_2(\mathrm{I}^k)^2\mathrm{II}^k/K^4 + d_3(\mathrm{II}^k/K^2)^2 + d_4\mathrm{I}^k\mathrm{III}^k/K^4 + \cdots\} \tag{52}$$

An approximate expression for the right side of (50) is now derived, based on the assumed smallness of $\mathbf{S}^k/K$. Letting $\beta = P^k/K$ and $\xi \ll 1$ be the magnitude of the largest term of $\mathbf{S}^k/K$, approximate expressions for the stress invariants are

$$\begin{aligned} \mathrm{I}^k &= -3K\beta \\ \mathrm{II}^k &= K^2[3\beta^2 + o(\xi)] \\ \mathrm{III}^k &= -K^3[\beta^3 + o(\xi)]. \end{aligned} \tag{53}$$

Substituting (53), (52), (51) into (50) and neglecting terms which are $o(\xi)$, we find

$$\mathbf{C} = \mathbf{I}p(\beta) + q(\beta)\mathbf{S}^k/K, \tag{54}$$

where

$$p(\beta) = a_1 - 2b_1\beta + (27c_1 - 9c_2 + c_3)\beta^2 + (-4d_1 + 18d_2 + 12d_3 - 21d_4)\beta^3 \tag{55a}$$

$$q(\beta) = b_2 - (3c_2 + 5c_3)\beta + (9d_2 + 6d_3 + 15d_4)\beta^3 \tag{55b}$$

Suppose we give the elastic deformation tensor $\mathbf{F}^e$ the representation

$$\mathbf{F}^e = \mathbf{R}^e(\delta\mathbf{I}+\mathbf{H}), \qquad \mathrm{tr}[\mathbf{H}] = 0, \tag{56}$$

where δ is a scalar function and $\mathbf{H}$ is small. The form (56) is compatible with our assumption that pure elastic deformation is a small deviation from a uniform dilation. An approximate expression for $\mathbf{C}$ is then

$$\mathbf{C} \equiv (\mathbf{F}^e)^T\mathbf{F}^e = \delta^2\mathbf{I}+2\delta\mathbf{H}+o(H). \tag{57}$$

The function q is now reduced to a single constant by making the following assumption: The force required to induce a small shear strain in an element of material under arbitrary dilatation is independent of the amount of volumetric strain. In terms of the true stress deviator $\mathbf{S}_1$, the supposition implies

$$\delta^2\mathbf{S} = 2\mu\mathbf{R}^e\mathbf{H}(\mathbf{R}^e)^T \tag{58}$$

where μ is the constant (perhaps temperature dependent) shear modulus. Comparison of (58) with (57) implies

$$q(\beta) \equiv K/\mu. \tag{59}$$

For comparison with Murnaghan's work on stress–strain relations [17], we note that the measure of dilation δ used here is related to his measure e through $\delta^2 = 1-2e$.

It remains to differentiate (54) to obtain $\underset{\sim}{\mathbf{K}}^e$ under the assumptions being employed here. It is convenient to carry out the differentiation treating $\mathbf{C}$ as a function of the eight independent components of $\mathbf{S}^k/K$ and β, rather than the nine components of $\mathbf{T}^k/K$, and differentiating by the chain rule

$$\underset{\sim}{\mathbf{K}}^e = \rho_0\frac{\partial\mathbf{C}}{\partial\mathbf{T}^k} = \rho_0\left\{\mathrm{tr}\left[\frac{\partial\mathbf{C}}{\partial\mathbf{S}^k}\frac{\partial\mathbf{S}^k}{\partial\mathbf{T}^k}\right]+\frac{\partial\mathbf{C}}{\partial\beta}\frac{\partial\beta}{\partial\mathbf{T}^k}\right\}. \tag{60}$$

The quantities β and $\mathbf{S}^k$ are given in terms of $\mathbf{T}^k$ by

$$\beta = -\frac{1}{3K}\mathrm{tr}[\mathbf{T}^k],$$

$$\mathbf{S}^k = \mathbf{T}^k-\frac{1}{3}\mathbf{I}\,\mathrm{tr}[\mathbf{T}^k].$$

The result of carrying out the differentiation indicated in (60) is

$$\underset{\sim}{\mathbf{K}}^e = -\rho_0\left(\frac{2}{3K}\frac{\mathrm{d}p}{\mathrm{d}\beta}+\frac{1}{3\mu}\right)\mathbf{I}\times\mathbf{I}+\rho_0\frac{1}{2\mu}\underset{\sim}{\mathbf{I}}, \tag{61}$$

where, in index notation,

$$(\mathbf{I}\times\mathbf{I})^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu} = \delta^{\alpha}{}_{\beta}\delta^{\mu}{}_{\nu},$$

$$(\underset{\sim}{\mathbf{I}})^{\alpha\beta}{}_{\mu\nu} = \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}+\delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu}. \tag{62}$$

The response function $\underset{\sim}{\mathbf{K}}^e$ is expressed as a zeroth order function in $\mathbf{S}^k/K$ because $\mathbf{C}$ was only first order and the order to which expansions may be accurately carried is reduced

by differentiation. That (61) represents the most general zeroth order expansion in $\mathbf{S}^k/K$ can be seen by considering (41), which is the most general expansion of a fourth order, isotropic tensor function of a second order tensor. If the argument tensor is diagonal, as it is for $\mathbf{S}^k/K = 0$, the only terms remaining are those with the form exhibited in (61). Letting b_1 be an appropriate function of Poisson's ratio, (61) can be reduced to the response function of linear elasticity.

The consequences of some assumptions concerning the plastic flow are now investigated. The objective is to reduce the complicated response function given in (41) to a simpler form. Consistent with experimental evidence from tests on most metals, we assume our material to be stable in the small, and to undergo deformations which are deviatoric and independent of hydrostatic pressure.

For large deformation problems, we take the generalization of Drucker's stability postulate [14] to be

$$\mathrm{tr}[\mathbf{D}^s\mathbf{B}] \geq 0, \tag{63}$$

where, as in (41)

$$\mathbf{B} \equiv \overset{\circ}{(\mathbf{T}^r/\rho^r)} + \sum_i \mathbf{M}_i \dot{\alpha}_i. \tag{64}$$

In (63), equality holds only when $\mathbf{B} \equiv 0$. The constitutive equation (38) then implies the following condition on $\mathbf{K}^s$, for any $\mathbf{B}$,

$$\mathrm{tr}[\underset{\sim}{\mathbf{K}}^s[\mathbf{B}]\mathbf{B}] \geq 0. \tag{65}$$

That is, $\underset{\sim}{\mathbf{K}}^s$ is a positive operator on the space of second order tensors. Let γ_k be the nine eigenvalues of $\underset{\sim}{\mathbf{K}}^s$, and let $\mathbf{r}^k$ and $\mathbf{l}^k$ represent the corresponding right and left eigenvectors. Then, with the set $\mathbf{l}^k$ as basis in the nine dimensional space, $\mathbf{D}^s$ has the representation

$$\mathbf{D}^s = \sum_{k=1}^{9} \gamma_k \,\mathrm{tr}[\mathbf{r}^k\mathbf{B}]\mathbf{l}^k.$$

We now assume that the kth component of stretching depends on $\mathbf{B}$ only through the component of $\mathbf{B}$ in the kth direction. Then $\mathbf{r}^k = \mathbf{l}^k$, and $\underset{\sim}{\mathbf{K}}^s$ is symmetric, that is

$$K^s{}_{\alpha\beta\mu\nu} = K^s{}_{\mu\nu\alpha\beta}. \tag{66}$$

Relation (66) generalizes the condition in the theory of infinitesimal plastic deformation that the strain increment depends only on the component of the stress increment in the direction of the strain increment. The symmetry property (66) imposes the condition

$$\Lambda_2 = \Lambda_4, \qquad \Lambda_3 = \Lambda_7, \quad \Lambda_6 = \Lambda_8$$

on the scalars of (41).

Next it is assumed that $\mathbf{D}^s$ is independent of the hydrostatic component of $\mathbf{T}^r$. Thus, we replace $\mathbf{T}^r$ in the constitutive relation by $\mathbf{S}^r = \mathbf{T}^r + \underset{\sim}{\mathbf{I}}P^r$, where $P^r = -\mathrm{tr}(T^r)/3$. Calculations are also simplified if we assume $\mathbf{M}_i$ to be proportional to $\mathbf{S}^r$, so this additional assumption is made. The latter assumption is consistent with the notion of effective stress discussed in relation to (40). Under these assumptions $\mathrm{tr}(\mathbf{B}) = 0$ and the scalars $\Lambda_1, \Lambda_4, \Lambda_7$ have been eliminated from (41).

We now limit ourselves exclusively to the form of the flow rule given in (40) where the effective stress is the actual stress multiplied by a single scalar. That is, in (41) we set $\mathbf{B} = \sigma^{-1}\overset{\circ}{\overline{\sigma \mathbf{S}^r/\rho^r}}$. The condition that the plastic deformation be deviatoric is satisfied by requiring that $\mathrm{tr}[\mathbf{D}^s] = 0$, along with the initial condition $\rho^r = \rho_0$. Imposing this condition on the remaining terms of (41), and observing that $-\frac{1}{2}\mathrm{tr}[(\mathbf{S}^r)^2]$ is the second principal invariant of $\mathbf{S}^r$, say II^r, we find that

$$3\Lambda_2 - 2\Lambda_8 \mathrm{II}^r/(\rho^r)^2 + 2\Lambda_{11} = 0 \tag{67a}$$

$$3\Lambda_3 - 2\Lambda_9 \mathrm{II}^r/(\rho^r)^2 + 2\Lambda_{12} = 0. \tag{67b}$$

Equations (67) allow us to eliminate two more of the set of coefficients Λ_k.

The yield function (42) is written in the special form

$$f(\sigma \mathbf{T}^r/\rho^r) = 0, \tag{68}$$

which is interpreted as a hypersurface in stress space. As in the theory of infinitesimal plasticity [14], the stability postulate leads directly to convexity of the yield surface and normality of the deformation rate to this surface. In terms of the yield function, the normality condition is expressed as

$$\mathbf{D}^s = \frac{\Phi}{\sigma}\frac{\partial f}{\partial \mathbf{T}^r} \tag{69}$$

where Φ is a scalar function of stress and σ which is homogeneous of degree one in $\overset{\circ}{\overline{\sigma \mathbf{T}^r}}$. The symmetry condition (66) then automatically requires that

$$\Phi = \sigma\Omega\,\mathrm{tr}\left[\frac{\partial f}{\partial \mathbf{T}^r}\overset{\circ}{\overline{\sigma \mathbf{T}^r}}\right], \tag{70}$$

where Ω is a scalar function of stress and σ. The flow rule then takes on the familiar form

$$\mathbf{D}^s = \Omega\frac{\partial f}{\partial \mathbf{T}^r}\,\mathrm{tr}\left[\frac{\partial f}{\partial \mathbf{T}^r}\overset{\circ}{\overline{\sigma \mathbf{T}^r}}\right]. \tag{71}$$

Since the assumed flow rule (40) is an isotropic one, we might wish to associate an isotropic yield function with it. That is, we assume f to depend only on the invariants of the effective stress. Incorporating the assumption that hydrostatic pressure does not influence yield, the yield function becomes

$$f(J_2, J_3) = 0, \tag{72}$$

where

$$J_2 = -\tfrac{1}{2}(\sigma/\rho^r)^2\,\mathrm{tr}[(\mathbf{S}^r)^2], \qquad J_3 = \tfrac{1}{3}(\sigma/\rho^r)^3\,\mathrm{tr}[(S^r)^3]. \tag{73}$$

The gradient in (71) is calculated to be

$$\frac{\partial f}{\partial \mathbf{T}^r} = \frac{\sigma}{\rho^r}\left\{-\frac{\partial f}{\partial J_2}\mathbf{S}^r\frac{\sigma}{\rho^r} + \frac{\partial f}{\partial J_3}\left[\left(\mathbf{S}^r\frac{\sigma}{\rho^r}\right)^2 + \frac{2}{3}J_2\mathbf{I}\right]\right\}. \tag{74}$$

We can now show that of the twelve scalar functions appearing in the response function (41), only two remain. At this point the remaining independent terms are Λ_k, $k = 5, 6, 9, 10, 11, 12$. The separability of $\underset{\sim}{\mathbf{K}}^s$ required to write (69) implies that

$$\Lambda_{10} = \Lambda_{11} = \Lambda_{12} = 0$$

$$\Lambda_6 = \Lambda_5^{\frac{1}{5}}\Lambda_9^{\frac{1}{5}}.$$

The resulting form of (41) is

$$\underset{\sim}{\mathbf{K}}^s[\mathbf{B}] = \{\tfrac{2}{3}\mathrm{II}^r\Lambda_9^{\frac{1}{5}}\mathbf{I}/(\rho^r)^2 + \Lambda_5^{\frac{1}{5}}(\mathbf{S}^r/\rho^r) + \Lambda_9^{\frac{1}{5}}(\mathbf{S}^r/\rho^r)^2\}\,\mathrm{tr}[\{\tfrac{2}{3}\mathrm{II}^r\Lambda_9^{\frac{1}{5}}\mathbf{I}/(\rho^r)^2 + \Lambda_5^{\frac{1}{5}}(\mathbf{S}^r/\rho^r) + \Lambda_9^{\frac{1}{5}}(\mathbf{S}^r/\rho^r)^2\}\mathbf{B}]. \tag{75}$$

Comparing (75) with (74), we can see that (75) and (71) have exactly the same form if we make the identification

$$-\frac{\sigma^2}{\rho^r}\frac{\partial f}{\partial J_2} = (\Lambda_5/\Omega)^{\frac{1}{5}},$$

$$\frac{\sigma^3}{\rho^r}\frac{\partial f}{\partial J_3} = (\Lambda_9/\Omega)^{\frac{1}{5}}. \tag{76}$$

As a particular case, we assume $\partial f/\partial J_3 = 0$, $\partial f/\partial J_2 = 1$ and $\Omega = \Omega(J_2)$. The flow rule reduces to the usual Hencky–Mises law

$$\mathbf{D}^s = \Omega\left(\frac{\sigma}{\rho^r}\right)^4 \mathrm{tr}[\overset{\circ}{\overline{\mathbf{S}^r}}\sigma\mathbf{S}^r]\frac{\mathbf{S}^r}{\rho^r},$$

with a Mises yield criterion.

CONCLUSION

The primary purpose of this paper was to demonstrate that, under assumptions which are usually valid for most metals undergoing large strain, high velocity deformation, very general constitutive laws could be reduced to a tractable form for multi-dimensional deformation processes. It was possible to carry through this development primarily because of the assumed weak interaction between elastic and plastic processes. By introducing the intermediate reference configuration, the aspects of elastic and plastic deformation could be discussed independently and in familiar terms. These separate results were then combined by observing additivity of the elastic and plastic stretching tensors, and by assuming a weak thermodynamic coupling between elastic and plastic effects.

Concerning applications, Murnaghan has shown that, for elastic deformation of the metal sodium, a theory involving volumetric strains only up to the second order often gave good agreement with experiment for pressures up to 100 kb.

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Абстракт—Предлагаются определяющие уравнения для описания поведения упруго-пластических материалов, подверженных большим деформациям. Со специальной кинематической точки зрения процессы упругой и пластической деформации можно рассматривать независимо. Это разделение приспособливается, также, упрощенной термодинамической теорией процесса деформации, Представляется определяющие уравнение в упругой области как уравнение скорости, после исследования интерпретации упругой изотропии, с точки зрения принятого подробного кинематического описания. Для описания же пластической деформации, предлагается уравнение скорости, которое проявляет независимость от скорости, при которой рассматривались предыдущие состояния. После приема общих зависимостей в подходящей форме рассматриваются некоторые упрощения основанные на физических предложениях. Эти физические предположения основанные на поведению металлов под влиянием больших напряжений, большой скорости нагрузки, таких как в задаче проникания. При учете этих условий, преобладают термоупругие эффекты и пластичность играет меньшую роль. Следовательно, несложная модель пластической деформации обычно достаточна. Предлагается анализ в элементарной /матричной/ записи. Этот анализ удовлетворяет произволным состояниям напряжений.